Inverse Solutions of the Richards Equation: A Comparison of Genetic and Levenberg-Marquardt Algorithms

Abbas El-Zein, George Chammoun and Leying Yang
School of Civil Engineering, University of Sydney, NSW 2006, Australia
(abbas.elzein@sydney.edu.au)

Abstract

Predicting water flow in partially-saturated soils, using the Richards equation, is important for a range of science and engineering problems. Unsaturated material properties, such as the water retention curve and the unsaturated hydraulic conductivity function, are usually expensive, time-consuming and difficult to measure experimentally. Inverse solutions of the Richards equation offer one alternative to experimental determination. However, the doubly-iterative process of inverting a highly non-linear partial differential equation such as the Richards equation leads to computationally expensive algorithms, and the choice of inverse solvers is critical in developing efficient inverse solution tools.

The paper assesses the performance of two popular inverse solution algorithms, the Levenberg-Marquardt Algorithm and Genetic Algorithms, and evaluates their comparative ability to invert the Richards equation accurately and efficiently. Tools are built using a forward-solver of the Richards equation based on the Finite-Element Method developed at the University of Sydney, and inverse algorithms available in Matlab. Using a simple 1D infiltration problem as a case, the inverse solver is assessed for a range of values for the difference between number of unknowns and number of observation points, as well as different levels of proximity of initial estimate to actual solution. For the case studied here, both algorithms generate accurate solutions; however, the Levenberg-Marquardt algorithm appears to be more computationally efficient.

Keywords: Richards Equation, Unsaturated Soils, Water Flow, Inverse Problems

Introduction

Predicting water flow in partially-saturated soils is important for a range of science and engineering problems, including water quality control, plant and soil science, groundwater protection, design of earth dams, foundations design, earth-dam construction and tunnelling, to name a few examples. The Richards equation is widely used to model the flow of water in partially-saturated soils. Based on Darcy's constitutive law and a mass conservation statement, it assumes no mass exchange occurs between the air and water phases in the soil, but incorporates the dependence of water content and hydraulic conductivity on water pressure, through two highly non-linear relationships, known as the water retention curve (WRC) and k_{unsat} function, respectively (e.g., Menziani et al., 2007).

When implemented within inverse solvers, the Richards equation can also be used to determine the WRC, the k_{unsat} function and the saturated hydraulic conductivity, i.e., soil material properties that are difficult and expensive to determine experimentally. However, inverse solvers of the Richards equation are likely to be computationally expensive because of the doubly iterative process involved:

a) iterations around the non-linearity of the forward solver, due to the dependence, mentioned above, of hydraulic conductivity and water content on pore pressure
and
b) multiple calls of the forward solver by the inverse solver before reaching the inverse solution.

Hence, the choice of algorithm for the inverse solver is critical in achieving convergence to the right solution while minimizing computational costs. A number of algorithms can be found in the literature for computing inverse solutions of non-linear problems (e.g., Haber et al., 2000). One of the most widely used techniques is the Levenberg-Marquardt algorithm (LMA), based on non-linear least-square optimisation (More, 1977). When the current iteration is far from the actual solution, the LMA behaves as a steepest-descent method with slow but robust convergence. However, as the actual solution is approached, the LMA becomes a faster Gauss-Newton algorithm (Lourakis, 2005).

More recently, genetic algorithms (GA), mirroring biological principles of natural selection and survival of the fittest, have been used to find solutions to inverse problems (Beasley et al., 1993). This family of methods assigns a fitness score to an initial 'population' of solutions, which is then used to generate a new population ('offspring') that is better 'adapted' to the problem at hand, until a solution of the inverse problem is reached. Both the LMA and GAs can be highly efficient and robust but do not always lead to the globally optimal solution. Hence, the suitability of either algorithms needs to be assessed for the specific problem at hand. Johari et al. (2006) used a GA to derive a formula for predicting water content in soils and found it reliable when comparing its predictions to experimental data. However, to the best of the authors' knowledge, no comparison has been made in the literature between the performances of the LMA and GAs in inverting a general form of the Richards equation.

The goal of this paper is to assess the efficiency of a GA in inverting the Richards equation, in comparison with the more conventional and widely-used LMA. The two solvers are built using Matlab tools as well as a finite-element solver of the Richards equations developed at the University of Sydney. The impact of two key factors is assessed a) the number of observation points N_o minus the number of unknowns N_u and b) the degree of proximity of the initial guess to the actual solution. Using a simple 1D infiltration problem as a based case, the two solvers are compared based on two criteria: a) the ability to converge to the right solution, and b) computational time required to achieve a given accuracy.

Methods

The 1D Richards equation can be written as follows:

$$\frac{\partial}{\partial y}\left(k_{yy}(\theta)\frac{\partial H}{\partial y}\right) = \left(\frac{\partial \theta}{\partial H}\right)\frac{\partial H}{\partial t} \quad (1)$$

where y [L] is the vertical component of a Cartesian coordinate system, t[T] is time, [L^3/L^3] is the volumetric water content of the soil, k_{yy} [$L.T^{-1}$] is the hydraulic conductivity of the soil and H [L] is the total hydraulic head with $H=h_p + y$ where h_p [L] is the pressure head (assuming the datum plane runs through the origin of axes); $h_p=p_w/(\rho g)$ where p_w [$M.L^{-1}.T^{-2}$] is the soil negative pore pressure, ρ [$M.L^{-3}$] is the density of water and g [$L.T^{-2}$] is the gravitational acceleration. Note that $\theta=nS$ where n is the porosity and S is the degree of saturation. The non-linearity of the Richards equation arises from the dependence of θ and k_{yy} on H (WRC and k_{unsat} function).

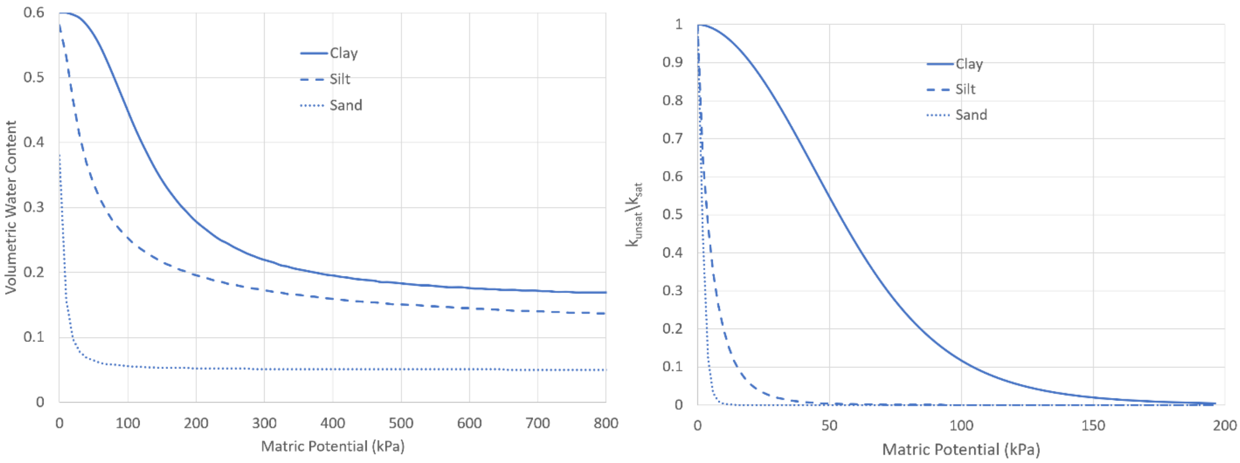
A widely-used WRC, and associated k_{unsat} function, first proposed by van Genuchten (1980), are adopted here and can be expressed in terms of pressure head:

$$\theta = \theta_r + (\theta_s - \theta_r)S_e \quad (2)$$

$$S_e = \left[1 + \left(\frac{h_p}{p_1}\right)^{p_3}\right]^{-p_2} \quad (3)$$

$$k_{yy} = k_{sat}\sqrt{S_e}\left[1 - \left(1 - S_e{}^{1/p_2}\right)^{p_2}\right]^2 \quad (4)$$

where θ_r and θ_s are the residual and saturated volumetric water contents, respectively; S_e is the effective degree of saturation; k_{sat} [$L.T^{-1}$] is the hydraulic conductivity at saturation; and p_1 [L], p_2 and p_3 are material property parameters with $p_3=1/(1-p_2)$. Figure 1 shows typical van Genuchten WRC and k_{unsat} functions for various soils.



	n	θ_r	θ_s	p_1 (m)	p_2	p_3
Clay	0.65	0.158	0.6	10.5	0.646	2.825
Silt	0.6	0.1	0.581	2.06	0.41	1.695
Sand	0.4	0.05	0.38	0.45	0.565	2.3

Figure 1. WRC and K_{unsat} van Genuchten functions for various soils; matric potential is the negative of pore pressure p_w; parameters shown in the table are partly based on Yang & You (2013).

CONFEM is a multi-purpose finite-element software for solving a range of water flow and contaminant migration problems, in saturated and partially-saturated media (El-Zein and Booker, 1999; El-Zein et al., 2005; El-Zein, 2008; El-Zein and Balaam, 2012). The code has been developed over the last decade at the University of Sydney along with a graphic interface, Soil Pollution Analysis System (SPAS), for building data and viewing results. A new capability for solving hydro-chemical problems in partially saturated media, including the Richards equation, has been recently added to the software. The algorithm for solving the Richards equation uses an iterative, Crank-Nicolson time-marching scheme. In the research reported in this paper, SPAS-CONFEM was used as the forward solver, called from the Matlab inversion tool.

The matlab function 'ga', used for the genetic algorithm, requires a user-defined objective function in the form of the sum of the squared residuals:

$$\sum_{i=1}^{N}\left(d_{obs,i} - d_{mod,i}\right)^2 \quad (5)$$

where $d_{obs,i}$ is the value of the observed data, $d_{mod,i}$ is the value of the SPAS-CONFEM modelled data and N is the number of observation data points. Least-square function 'lsqnonlin', with option 'levenberg-marquardt', was used for the LMA. The function accepts a user-defined array of the residuals, F, and calculates the sum of the residuals, as shown in equations (6) and (7), respectively:

$$F = \begin{bmatrix} d_{obs,1} - d_{mod,1} \\ d_{obs,2} - d_{mod,2} \\ \cdots \\ d_{obs,i} - d_{mod,i} \\ \cdots \\ d_{obs,N} - d_{mod,N} \end{bmatrix} \quad (6)$$

$$\sum_{i=1}^{N} F_i^{\,2} \quad (7)$$

The genetic algorithm used here can be classified as a standard genetic algorithm (Beasley et al., 1993). Although more complex genetic algorithms may have superior performance (e.g., parallel genetic algorithms), the aim of the research was to assess widely-available standard procedures (Sivanandam and Deepa 2008, pp. 107,).

Results

The performance of the algorithms was tested in two stages. In the first stage, the accuracy of the forward solver SPAS was evaluated by comparing its predictions to analytical solutions as well as those of a commercial water flow software SEEP/W, for a range of problems in 1D, 2D Cartesian and 2D axisymmetric. Good agreement was found between SPAS and analytical and numerical predictions. This is illustrated by Figure 3 which shows matric potential versus depth (matric potential ψ=-p_w) at three different times.

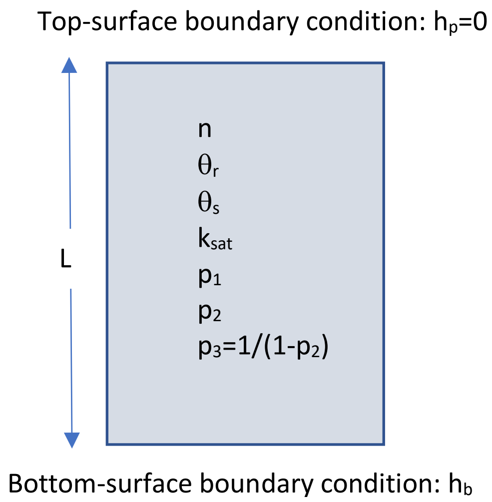


Figure 2. Infiltration Problem Studied

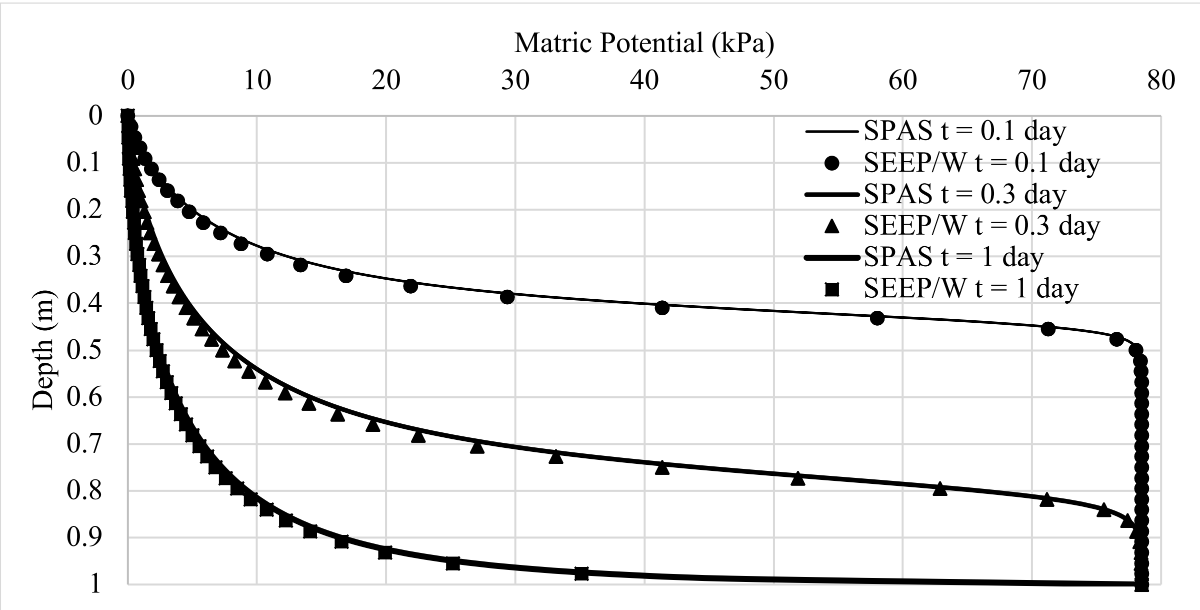


Figure 3. Comparison of predictions of SPAS to predictions from SEEP/W (L=1m, h_b=-8m, k_{sat}=1.16x10^{-6}, n=0.4, θ_r=0.186, θ_s=0.383, p_1=24m, p_2=0.346, p_3=1.53).

In the second stage, the problem shown in Figure 2 was analysed in steady-state, using forward SPAS, and pressure heads at selected nodes were used as observation points in subsequent inverse analyses. The following parameters were used: L=10m, h_b=-20m, k_{sat}=10^{-8}, n=0.45, θ_r=0.15, θ_s=0.4, p_1=25m,

p_2=0.6, p_3=2.5. Tables 1 and 2 compare the performances of the LMA and GA algorithms. The error at each observation point (difference between pressure head from forward and inverse analyses) was calculated and both the maximum and average of those errors are shown in Tables 1 and 2. The LMA shows excellent accuracy in all cases except when the number of observation points is 2 with two unknows to be determined (p_1 and p_2). The GA algorithm yields results within a few percentages of the exact solution, even when only three observation points are available. However, overall, the LMA performs better in terms of accuracy and CPU time. Furthermore, it can be seen from Table 2 that, under the conditions studied in this paper, the LMA appears to be more robust that the GA when the initial guess is made further removed from the exact solution.

Table 1. Comparison of performances of GA and LMA under different numbers of observation points (unknowns are p_1 and p_2 with initial guesses p_1=40m, p_2=0.5)

	Genetic Algorithm			Levenberg-Marquardt Algorithm		
Number of Observation Points	**Max Error**	**Average Error**	**CPU Time (min)**	**Max Error**	**Average Error**	**CPU Time (min)**
61	3%	2%	1.96	0%	0%	0.76
31	7%	5%	3.35	0%	0%	0.86
21	3%	3%	3.34	0%	0%	0.91
11	7%	6%	2.26	0%	0%	0.35
3	9%	8%	3.02	60%	44%	0.74

Table 2. Comparison of Performances of GA and LMA under Different Initial Guesses (31 observation points, two unknowns p_1 and p_2; percentage differences between initial guesses and exact solution are shown in brackets)

Initial Guesses		Genetic Algorithm			Levenberg-Marquardt Algorithm		
p_1 (m)	**p_2**	**Max Error**	**Average Error**	**CPU Time (min)**	**Max Error**	**Average Error**	**CPU Time (min)**
35 (40%)	0.8 (33%)	12%	10%	3.24	0%	0%	0.31
30 (20%)	0.7 (17%)	24%	24%	5.53	0%	0%	0.52
25 (0%)	0.6 (0%)	5%	3%	2.79	0%	0%	0.07
20 (-20%)	0.5 (-17%)	5%	4%	4.74	0%	0%	0.35
15 (-40%)	0.4 (-17%)	15%	12%	5.17	32%	28%	0.8

Conclusions

The paper has assessed the relative merits of two algorithms for inverting the Richards equation. It has shown that both algorithms can back-calculate water retention parameters of soils with good accuracy, although the LMA performs better in terms of accuracy and CPU time. Observation points from a steady-state analysis of one simple infiltration problem for one soil were used for the analyses here. However, the findings are must be seen as preliminary and the work needs to be broadened to investigate the effects of key parameters in the genetic algorithm (e.g., range, size of initial populations, means of generating initial populations), other water flow problems with different boundary conditions, different types of soils with different material properties, as well as the effect of using observation points from time-dependent analyses rather than steady-state ones.

Several attempts have been made in the literature to infer water retention properties of soil from infiltration tests in the lab (e.g., Garnier et al., 1997; Bruckler et al., 2002; Schelle et al., 2011; Nasta et al., 2011). However, these tests are usually based on boundary conditions that are difficult to maintain or measure in the laboratory and their accuracy is sometimes questionable. A useful extension of the work reported in this paper is to investigate whether simple infiltration tests with simple boundary conditions can be used to back-calculate water retention properties of different types of soils. This work is currently in progress.

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